

Boshlang'ich funksiya va aniqmas integral

12.1.1- ta'rif. Agar D sohada $f(x)$ funksiya aniqlangan bo'lib, bu sohada hosilasi $f(x)$ ga teng bo'lgan $F(x)$ funksiya mavjud bo'lsa, u $f(x)$ funksiyaning *boshlang'ich funksiyasi* deyiladi.

Demak, ta'rif bo'yicha $F'(x)=f(x)$ bo'lsa, $F(x)$ funksiya $f(x)$ ning boshlang'ich funksiyasidir.

Masalan, $f(x)=x$ uchun $F(x)=\frac{x^2}{2}$ boshlang'ich funksiyadir, chunki,

$$F'(x) = \left(\frac{x^2}{2} \right)' = \frac{1}{2} \cdot 2x = x = f(x).$$

Shu misolda $F_1(x) = \frac{x^2}{2} + C$, bu yerda C -qandaydir o'zgarmas son, funksiyani qarasak,

$$F_1'(x) = \left(\frac{x^2}{2} + C \right)' = \left(\frac{x^2}{2} \right)' + C' = x + 0 = f(x)$$

bo'lib, u ham $f(x)$ ning boshlang'ich funksiyasi bo'lar ekan. $C \in \mathbb{R}$ ekanligidan bu berilgan funksiyaning boshlang'ich funksiyalarining cheksiz ko'pligi kelib chiqadi. Bu xulosa umumiy bo'lib, quyidagi teorema o'rinalidir.

12.1.1-teorema. Agar $(a;b)$ oraliqda aniqlangan $f(x)$ funksiyaning shu oraliqda $F(x)$ boshlang'ich funksiyasi mavjud bo'lsa, uning bu oraliqda cheksiz ko'p boshlang'ich funksiyalari mavjud bo'lib, ular bir-biridan biror o'zgarmas qo'shiluvchiga farq qiladi.

Isbot. Aytaylik, $(a;b)$ da $F'(x)=f(x)$ bo'lsin. Agar $F(x)+C$, bu yerda $C \in \mathbb{R}$ o'zgarmas, yig'indini olsak, u ham $(a;b)$ da aniqlangan va $(F(x)+C)' = F'(x) + C' = f(x) + 0 = f(x)$, ya'ni $f(x)$ ning boshlang'ich funksiyasi ekanligi va $C \in \mathbb{R}$ ixtiyoriy o'zgarmas ekanligidan ularning cheksiz ko'pligi kelib chiqadi.

Endi, faraz qilaylik, $(a;b)$ oraliqda $F_1(x)$ va $F_2(x)$ lar $f(x)$ funksiyaning qandaydir ikkita boshlang'ich funksiyalari bo'lsin, u vaqtda, $[F_1(x) - F_2(x)]' = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$, bundan $(a;b)$ oraliqda $F_1(x) - F_2(x) = C_0$ - o'zgarmas ekanligi, ya'ni $F_1(x)$ va $F_2(x)$ lar bir-biridan C_0 o'zgarmas ko'shiluvchiga farq qilishi kelib chiqadi, ya'ni $F_1(x) = F_2(x) + C_0$ dir.

Teorema isbotlandi.

12.1.2-ta'rif. Agar D sohada aniqlangan $f(x)$ funksiyaning boshlang'ich funksiyasi $F(x)$ mavjud bo'lsa, boshlang'ich funksiyalarining $\{F(x)+C : C \in \mathbb{R}\}$ to'plamini $f(x)$ funksiyaning *aniqmas integrali* deyiladi va $\int f(x)dx$ kabi belgilanadi.

Bu yerda $\int$ -integral belgisi, $f(x)$ -integral osti funksiyasi, $f(x)dx$ -integral osti ifodasi, x - integral o'zgaruvchisi deb yuritiladi.

Ta'rifga asoslangan holda $\int f(x)dx = F(x) + C$ ko'rinishda yozish qabul qilingan bo'lib, bu yerda $F'(x)=f(x)$, $C \in \mathbb{R}$ ixtiyoriy o'zgarmasdir.

Masalan, yuqorida ko'rilgan misolni $\int x dx = \frac{x^2}{2} + C$ kabi yozish mumkin.

Berilgan funksiyaning boshlang'ich funksiyasini, ya'ni aniqmas integralini topish jarayoni uni integrallash deb yuritiladi. Yuqoridagi ta'riflardan ko'rinadiki, integrallash differensiallashga teskari amaldir. Buni hisobga olsak, differensiallashning asosiy qoidalaridan integrallash uchun quyidagi asosiy xossalar kelib chiqadi:

$$1^0. \int f(x)dx = F(x), \quad d\left[\int f(x)dx\right] = f(x)dx;$$

$$2^0. \int du(x) = u(x) + C, \quad C - \text{ixtiyoriy o'zgarmas};$$

$$3^0. \int Af(x)dx = A \int f(x)dx, \quad A - \text{o'zgarmas ko'paytuvchi};$$

$$4^0. \int \left[\sum_{i=1}^n A_i f_i(x) \right] dx = \sum_{i=1}^n A_i \int f_i(x)dx, \quad A_i (i = \overline{1, n}) - \text{o'zgarmaslar};$$

$$5^0. \int f(x)dx = F(x) + C \text{ bo'lib, } x \text{ biror } D \text{ sohada o'zgarganda } u(x) -$$

differensiallanuvchi funksiya hamda uning qiymati $f(x)$ ning aniqlanish sohasiga tegishli bo'lsa, $\int f(u(x))du(x) = F(u(x)) + C$ bo'ladi.

Bu yerda $du(x) = u'(x)dx$ ekanligini eslash lozimdir.

Hosila jadvali hamda integrallashning yuqoridagi xossalari asosida olinadigan quyidagi integrallar jadvalini keltiramiz.

Integrallar jadvali

$$1. \int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad -1 \neq \alpha \in R$$

$$2. \int \frac{dx}{x} = \ln |x| + C$$

$$3. \int a^x dx = \frac{a^x}{\ln a} + C, \quad 1 \neq a \in R^+; \quad \int e^x dx = e^x + C$$

$$4. \int \sin x dx = -\cos x + C$$

$$5. \int \cos x dx = \sin x + C$$

$$6. \int \operatorname{tg} x dx = -\ln |\cos x| + C$$

$$7. \int \operatorname{ctg} x dx = \ln |\sin x| + C$$

$$8. \int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$

$$9. \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$10. \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C \quad (= -\arccos x + C)$$

$$11. \int \frac{dx}{1+x^2} = \operatorname{arctg} x + C \quad (= -\operatorname{arcctg} x + C)$$

$$12. \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C \quad (= -\arccos \frac{x}{a} + C), \quad 0 \neq a \in R$$

$$13. \int \frac{dx}{a^2+x^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C \quad \left(= -\frac{1}{a} \operatorname{arcctg} \frac{x}{a} + C \right), \quad 0 \neq a \in R$$

$$14. \quad \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C, \quad 0 \neq a \in R$$

$$15. \quad \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C, \quad 0 \neq a \in R$$

$$16. \quad \int \frac{dx}{\sin x} = \ln \left| \operatorname{tg} \frac{x}{2} \right| + C$$

$$17. \quad \int \frac{dx}{\cos x} = \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C$$

Bulardan 1-5, 8-11- formulalar hosila jadvalidan bevosita kelib chiqadi. Masalan, 2-formulani olsak, $x > 0$ bo'lgan holda $(\ln x)' = \frac{1}{x} \Rightarrow \int \frac{dx}{x} = \ln x + C$ kelib chiqadi. $x < 0$

bo'lgan holni qaralsa, $|x| = -x$ bo'lganligi sababli,

$$(\ln |x|)' = (\ln(-x))' = \frac{1}{-x} \cdot (-1) = \frac{1}{x} \Rightarrow \int \frac{dx}{x} = \ln |x| + C \text{ ni olamiz}$$

6,7 va 12,13- formulalar integrallashning yuqorida keltirilgan 5^0 -xossasidan va integrallar jadvalidan foydalangan holda olinadi. Masalan, 5^0 -xossa va integrallar jadvalining 2-formulasiga asosan

$$\int \operatorname{tg} x dx = \int \frac{\sin x}{\cos x} dx = - \int \frac{d(\cos x)}{\cos x} = - \ln |\cos x| + C$$

integrallar jadvalining 6-formulasini olamiz.

14 – formula quyidagicha olinadi:

$$\begin{aligned} \int \frac{dx}{x^2 - a^2} &= \int \frac{1}{2a} \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx = \frac{1}{2a} \left(\int \frac{d(x-a)}{x-a} - \int \frac{d(x+a)}{x+a} \right) = \\ &= \frac{1}{2a} (\ln |x-a| - \ln |x+a|) + C = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C. \end{aligned}$$

15-17- formulalarni keyinroq keltirib chiqaramiz.

12.2. Integrallash usullari

12.2.1. Bevosita integrallash. Bunda integrallar jadvali va integrallashning asosiy xossalariidan foydalaniladi.

1-misol. $\int \left(\frac{x-1}{x+1} + e^x - \sin x + \frac{x}{\sqrt{x^2+1}} \right) dx$ ni toping,

Yechish:

$$\begin{aligned} \int \left(1 - \frac{2}{x+1} + e^x - \sin x + \frac{x}{\sqrt{x^2+1}} \right) dx &= \int dx - 2 \int \frac{dx}{x+1} + \int e^x dx - \int \sin x dx + \\ &+ \int \frac{xdx}{\sqrt{x^2+1}} = x - 2 \int \frac{d(x+1)}{x+1} + e^x - (-\cos x) + \\ &+ \int \frac{d(x^2+1)}{2\sqrt{x^2+1}} = x - 2 \ln |x+1| + e^x + \cos x + \sqrt{x^2+1} + C. \end{aligned}$$

Bu yerda $\int \frac{dx}{2\sqrt{x}} = \sqrt{x} + C$ $\left((\sqrt{x})' = \frac{1}{2\sqrt{x}} \right)$ dan ham foydalanildi.

2-misol. $P_n(x) = \sum_{i=0}^n a_i x^{n-i}$ ko'phadni integrallang.

Yechish:
$$\int P_n(x) dx = \int \sum_{i=0}^n a_i x^{n-i} dx = \sum_{i=0}^n a_i \int x^{n-i} dx = \sum_{i=0}^n a_i \frac{x^{n-i+1}}{n-i+1} + C$$
$$= \sum_{i=0}^n \frac{a_i}{n-i+1} x^{n-i+1} + C.$$

Demak, n -darajali ko'phadning aniqmas integrali $n+1$ -darajali ko'phaddan iborat bo'lar ekan.

Masalan,
$$\int (x^2 - 2x + 3) dx = \int x^2 dx - 2 \int x dx + 3 \int dx = \frac{x^3}{3} - 2 \cdot \frac{x^2}{2} + 3x + C =$$
$$= \frac{x^3}{3} - x^2 + 3x + C.$$

12.2.2. Bo'laklab integrallash. $d(uv) = vdu + u dv$ ekanligidan

$$\int d(uv) = \int (vdu + u dv) \Rightarrow uv = \int vdu + \int u dv \Rightarrow$$
$$\int u dv = uv - \int vdu. \quad (12.2.1)$$

Bu bo'laklab integrallash formulasi deb yuritiladi.

3-misol. $\int \ln x dx$ topilsin.

Yechish: $u = \ln x, dv = dx \Rightarrow du = \frac{dx}{x}, v = x.$ (12.2.1) formulaga asosan:

$$\int \ln x dx = (\ln x)x - \int x \frac{dx}{x} = x \ln x - \int dx = x \ln x - x + C = x(\ln x - 1) + C.$$

4-misol. $\int x^2 \cos x dx$ topilsin.

Yechish. $u = x^2, dv = \cos x dx \Rightarrow du = 2x dx, v = \sin x.$

$$\int x^2 \cos x dx = x^2 \sin x - \int \sin x \cdot 2x dx = x^2 \sin x - 2 \int x \sin x dx$$

oxirgi integralni ham bo'laklaymiz:

$$u_1 = x, dv_1 = \sin x dx \Rightarrow du_1 = dx, v_1 = -\cos x$$

$$\int x^2 \cos x dx = x^2 \sin x - 2(x \cdot (-\cos x) - \int (-\cos x) \cdot dx) = x^2 \sin x + 2x \cos x - 2 \int \cos x dx =$$
$$= x^2 \sin x + 2x \cos x - 2 \sin x + C = (x^2 - 2) \sin x + 2x \cos x + C.$$

5-misol. $\int e^x \sin x dx$ topilsin.

Yechish. $u = \sin x, dv = e^x dx \Rightarrow du = \cos x dx, v = e^x.$

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x \cdot dx$$

oxirgi integralni ham bo'laklaymiz:

$$u_1 = \cos x, dv_1 = e^x dx \Rightarrow du_1 = -\sin x dx, v_1 = e^x$$

$$\int e^x \sin x dx = e^x \sin x - (e^x \cos x - \int e^x (-\sin x dx)) = e^x \sin x - e^x \cos x + \int e^x \sin x dx + C_1.$$

Bu yerda $\int e^x \sin x dx = e^x (\sin x - \cos x) - \int e^x \sin x dx$ bo'lib, u topilishi talab qilingan integralga nisbatan tenglamadir. Ya'ni

$$2 \int e^x \sin x dx = e^x (\sin x - \cos x) + C_1.$$

$$\text{Bundan } \int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

ni olamiz $\left(C = \frac{C_1}{2} \right)$.

12.2.3. Aniqmas integralda o'zgaruvchini almashtirish (o'rniga qo'yish) usuli.

1. Aytaylik, $f(x)$ funksiyaning biror D sohada aniqlangan va $f(x)$ boshlangich $\int f(x)dx = F(x) + C$ bo'lib, $x = \varphi(t)$ funksiya biror T sohada differentsiallanuvchi hamda t ning T sohaga tegishli qiymatiga $\varphi(t) \in D$ bo'lsin.

U holda integrallashning 5⁰- xossasiga ko'ra

$$\int f[\varphi(t)]d\varphi(t) = F(\varphi(t)) + C = F(x) + C$$

Endi, $d\varphi(t) = \varphi'(t)dt$ ekanligini hisobga olsak, $\int f[\varphi(t)]\varphi'(t)dt = F(x) + C$, ya'ni

$$\int f(x)dx = \int f[\varphi(t)] \cdot \varphi'(t)dt \quad (12.2.2)$$

ni olamiz. Bu *aniqmas integralda o'zgaruvchini almashtirish formulasi* deb yuritiladi.

2. Xuddi yuqoridagiga o'xshash $\int f[q(x)]q'(x)dx$ integral qaralayotgan bo'lsa, $t=q(x)$ almashtirish (o'rniga qo'yish) yordamida

$$\int f[q(x)] \cdot q'(x)dx = \int f(t)dt \quad (12.2.3)$$

ni olamiz. Bu *aniqmas integralda o'rniga qo'yish usuli* deb yuritiladi.

6-misol. $\int \sqrt{a^2 - x^2} dx$ ni toping ($a > 0$).

Yechish. $x = a \sin t$ almashtirish qilamiz. (12.2.2) formulaga asosan

$$\begin{aligned} \int \sqrt{a^2 - x^2} dx &= \int \sqrt{a^2 - a^2 \sin^2 t} \cdot a \cos t dt = a^2 \int \cos^2 t dt = \\ &= a^2 \int \frac{1 + \cos 2t}{2} dt = \frac{a^2}{2} \left(\int dt + \int \cos 2t dt \right) = \\ &= \frac{a^2}{2} \left(t + \frac{1}{2} \sin 2t \right) + C = \frac{a^2}{2} \left(t + \frac{1}{2} 2 \sin t \cos t \right) + C = \\ &= \frac{a^2}{2} \left(t \arcsin \frac{x}{a} + \frac{x}{a} \sqrt{1 - \frac{x^2}{a^2}} \right) + C = \frac{a^2}{2} \arcsin \frac{x}{a} + x \sqrt{a^2 - x^2} + C. \end{aligned}$$

$$\text{Demak, } \int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \arcsin \frac{x}{a} + x \sqrt{a^2 - x^2} + C.$$

Bu yerda $|x| \leq a \Rightarrow t \in \left[-\frac{\pi}{2}; \frac{\pi}{2} \right] \Rightarrow \cos t \geq 0$ ekanligi hisobga olindi.

7-misol. $\int \frac{\cos x dx}{\sqrt{\sin x}}$ ni toping.

Yechish: $\cos x dx = d \sin x$ ekanligidan (12.2.3) formulaga ko'ra

$$\int \frac{\cos x dx}{\sqrt{\sin x}} = \int \frac{d \sin x}{\sqrt{\sin x}} = \int \frac{dt}{\sqrt{t}} = 2 \int \frac{dt}{2\sqrt{t}} = 2\sqrt{t} + C = 2\sqrt{\sin x} + C.$$

Bu yerda $\sin x = t$ o'rniga qo'yishdan foydalandik.

8-misol. $\int \frac{dx}{\sqrt{a^2 - x^2}}$, ni toping ($a > 0$).

Yechish: $x = at \Rightarrow dx = a dt \Rightarrow \int \frac{2x}{\sqrt{a^2 - x^2}} = \int \frac{adt}{\sqrt{a^2 - x^2 t^2}} = \int \frac{dt}{\sqrt{1 - t^2}} =$
 $= \arcsin t + C = \arcsin \frac{x}{a} + C.$

Integrallar jadvalining 12-formulasini oldik.

9-misol. $\int \frac{dx}{a^2 + x^2}$ ni toping ($a > 0$).

Yechish: $x = at \Rightarrow dx = a dt \Rightarrow \int \frac{dx}{a^2 + x^2} = \int \frac{adt}{a^2 + a^2 t^2} = \frac{1}{a} \int \frac{dt}{1 + t^2} =$
 $= \frac{1}{a} \arctgt + C = \frac{1}{a} \arctg \frac{x}{a} + C.$

Integrallar jadvalining 13-formulasini oldik.

12.3. Ratsional funksiyani (kasrni) integrallash

Ratsional funksiya deganda ikkita ko'phadning nisbatidan iborat bo'lgan funksiyani tushunamiz. Ya'ni, agar $P_n(x)$ va $Q_m(x)$ lar mos ravishda n – va m – darajali ko'phadlar bo'lsa,

$$R(x) = \frac{Q_m(x)}{P_n(x)} \quad (12.3.1)$$

kasr ratsional funksiyadir. Ba'zan ratsional funksiya deyish o'rniga *ratsional kasr* atamasi ham ishlatiladi.

Agar $m < n$ bo'lsa (12.3.1) *to'g'ri ratsional kasr*, $m \geq n$ bo'lganda esa *noto'g'ri ratsional kasr* deb yuritiladi. Shuni ham eslatamizki, agar (12.3.1) noto'g'ri ratsional kasr bo'lsa, bo'lish amalini ko'phadlar ustida bajarib, uni butun qismi bo'lgan biror ko'phad bilan kasr qismi bo'lgan biror to'g'ri ratsional kasr yig'indisiga keltirish mumkin.

Endi, ratsional kasrni integrallash masalasiga kelsak, yuqoridagi mulohazalardan ko'rinadiki, ko'phad bilan to'g'ri ratsional kasrni integrallashga kelamiz. Ko'phadni integrallashni bilamiz. Endi to'g'ri ratsional kasrni integrallash masalasini qaraymiz. Algebra kursidan ma'lumki, haqiqiy koeffitsientli ko'phadni chiziqli hamda diskriminanti manfiy bo'lgan kvadrat uchhad ko'rinishidagi ko'paytuvchilarga ajratish mumkin. Shuni ham eslatamizki, bunday ko'paytuvchilar faqat chiziqli yoki faqat kvadrat uchhad ko'rinishda ham bo'lishi mumkin (11-bobga qarang). Umumiylikka ta'sir qilmagan holda, (12.3.1) da $m < n$ va $P_n(x)$ ko'phadning bosh koeffitsienti birga teng deb olamiz (agar 1 dan farqli bo'lsa, surat va maxrajni shu bosh koeffitsientga bo'lish bilan bunga erishish mumkin). U vaqtda, yuqorida aytilganlarga asosan, $P_n(x)$ ko'phad uchun

$$P_n(x) = \left(\prod_{i=1}^s (x - a_i)^{k_i} \right) \cdot \left(\prod_{j=1}^l (x^2 + p_j x + q_j)^{r_j} \right) \quad (12.3.2)$$

ko'paytuvchilarga yoyilma o'rinli bo'lib, bu yerda

$$\sum_{i=1}^s k_i + 2 \sum_{j=1}^l r_j = n,$$

$a_i (i = \overline{1; k})$ – lar $P_n(x)$ ning haqiqiy ildizlari, $p_j^2 - 4q_j < 0 (j = \overline{1; l})$ dir.

Agar (12.3.1) to'g'ri ratsional kasr bo'lib, uning maxraji (12.3.2) ko'rinishda ko'paytuvchilarga ajratilgan bo'lsa, uni

$$\frac{Q_m(x)}{P_n(x)} = \sum_{i=1}^s \sum_{\lambda=1}^{k_i} \frac{A_{i\lambda}}{(x-a_i)^\lambda} + \sum_{j=1}^l \sum_{\mu=1}^{r_j} \frac{B_{j\mu}x + C_{j\mu}}{(x^2 + p_jx + q_j)^\mu} \quad (12.3.3.)$$

ko'rinishdagi *sodda (elementar) kasrlarning* yig'indisi sifatida ifodalash mumkin [7]. (12.3.3) dagi $A_{i\lambda}$, $B_{j\mu}$ va $C_{j\mu}$ koeffitsientlar uning o'ng tomonini umumiy maxrajga keltirilgach, chap va o'ng tomon maxrajleri bir xilligidan suratlar tengligini e'tiborga olib, ikki ko'phadning aynan tengligi shartidan foydalanib olingan chiziqli tenglamalar sistemasidan topiladi. Buni quyidagi misollar ustida tushuntiramiz.

1-misol. $\frac{x-1}{x^3+1}$ kasrni sodda ratsional kasrlar yig'indisiga keltiring.

Yechish: Buning uchun $x^3+1=(x+1)(x^2-x+1)$ ekanligidan foydalanamiz. (12.3.3) ga asosan:

$$\begin{aligned} \frac{x-1}{x^3+1} &= \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}, \\ x-1 &= A(x^2-x+1) + (Bx+C)(x+1), \\ x-1 &= (A+B)x^2 + (-A+B+C)x + (A+C); \end{aligned}$$

mos koeffitsientlarni tenglashtirib,

$$\begin{cases} A+B=0, \\ -A+B+C=1, \\ A+C=-1 \end{cases}$$

sistemani olamiz. Uni yechib, $A=-\frac{2}{3}, B=\frac{2}{3}, C=\frac{1}{3}$ larga ega bo'lamiz.

Demak,

$$\frac{x-1}{x^3+1} = -\frac{2}{3(x+1)} + \frac{1}{3} \cdot \frac{2x-1}{x^2-x+1} = \frac{1}{3} \left(\frac{2x-1}{x^2x+1} - \frac{2}{x+1} \right).$$

2-misol. $\frac{1}{x^4+2x^3+x^2}$ ni sodda ratsional kasrlar yig'indisiga keltiring

Yechish: $x^4+2x^3+x^2=x^2(x^2+2x+1)=x^2(x+1)^2$. (12.3.3) ga asosan:

$$\begin{aligned} \frac{1}{x^4+2x^3+x^2} &= \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x+1} + \frac{A_4}{(x+1)^2}; \\ 1 &= A_1x(x+1)^2 + A_2(x+1)^2 + A_3x^2(x+1) + A_4x^2. \end{aligned}$$

Noma'lum koeffitsientlarni x ga ma'lum qiymatlar berish yo'li bilan ham topish mumkin bo'lib, ko'p hollarda bu qulaylik tug'diradi.

Oxirgida

$$x=0 \text{ desak, } 1=A_1 \cdot 0 + A_2 \cdot 1 + A_3 \cdot 0 + A_4 \cdot 0 \Rightarrow A_2=1;$$

$$x=-1 \text{ desak, } 1=A_1 \cdot 0 + A_2 \cdot 0 + A_3 \cdot 0 + A_4 \cdot 1 \Rightarrow A_4=1;$$

$$x=1 \text{ desak, } 1=4A_1+4A_2+2A_3+A_4 \Rightarrow 4A_1+2A_3=-4;$$

$$x=-2 \text{ desak, } 1=-2A_1+A_2-4A_3+4A_4 \Rightarrow -2A_1-4A_3=-4;$$

$$\begin{aligned} \begin{cases} 4A_1+2A_3=-4 \\ -2A_1-4A_3=-4 \end{cases} &\Rightarrow \begin{cases} -4A_1-2A_3=4 \\ A_1+2A_3=2 \end{cases} \Rightarrow \begin{cases} -3A_1=6 \\ A_3=-2-2A_1 \end{cases} \Rightarrow \begin{cases} A_1=-2 \\ A_3=2 \end{cases} \\ A_1 &=-2, \quad A_2=1, \quad A_3=2, \quad A_4=1. \end{aligned}$$

Demak, $\frac{1}{x^4 + 2x^3 + x^2} = -\frac{2}{x} + \frac{1}{x^2} + \frac{2}{x+1} + \frac{1}{(x+1)^2}$.

Shunday qilib, to'g'ri ratsional kasrni integrallash masalasi, (12.3.3) yoyilmaga ko'ra, integrallanishi jihatidan bir-biridan farq qiladigan quyidagi to'rt xil sodda (elementar) ratsional kasrlarni integrallash masalasiga keltiriladi:

$$\frac{A_0}{x-a}, \frac{A_0}{(x-a)^n}, \frac{B_0x+C_0}{x^2+px+q}, \frac{B_0x+C_0}{(x^2+px+q)^n};$$

bu yerda A_0, B_0, C_0, a, p va q lar berilgan sonlar: $2 \leq n \in N, p^2 - 4q < 0$ dir.

Bu sodda ratsional kasrlarni integrallaylik.

1) $\int \frac{A_0}{x-a} dx = A_0 \int \frac{d(x-a)}{x-a} = A_0 \ln |x-a| + C.$

2) $\int \frac{A_0}{(x-a)^n} dx = A_0 \int (x-a)^n d(x-a) = A_0 \frac{(x-a)^{-n+1}}{-n+1} + C = -\frac{A_0}{(n-1)(x-a)^{n-1}} + C, (2 \leq n \in N)..$

3)
$$\begin{aligned} \int \frac{B_0x+C_0}{x^2+px+q} dx &= \int \frac{\frac{B_0}{2}(2x+p) + \left(C_0 - \frac{B_0p}{2}\right)}{x^2+px+q} dx = \\ &= \frac{B_0}{2} \int \frac{(2x+p)dx}{x^2+px+q} + \left(C_0 - \frac{B_0p}{2}\right) \int \frac{dx}{\left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}} = \\ &= \frac{B_0}{2} \int \frac{d(x^2+px+q)}{x^2+px+q} + \left(C_0 - \frac{B_0p}{2}\right) \int \frac{d\left(x + \frac{p}{2}\right)}{\left(x + \frac{p}{2}\right)^2 + \left(\frac{\sqrt{4q-p^2}}{2}\right)^2} = \\ &= \frac{B_0}{2} \ln(x^2+px+q) + \frac{C_0 - \frac{B_0p}{2}}{\frac{\sqrt{4q-p^2}}{2}} \cdot \operatorname{arctg} \frac{x + \frac{p}{2}}{\frac{\sqrt{4q-p^2}}{2}} + C = \\ &= \frac{B_0}{2} \ln(x^2+px+q) + \frac{2C_0 - B_0p}{\sqrt{4q-p^2}} \operatorname{arctg} \frac{2x+p}{\sqrt{4q-p^2}} + C. \end{aligned}$$

4)
$$\begin{aligned} 2 \leq n \in N, \quad \int \frac{B_0x+C_0}{(x^2+px+q)^n} &= \frac{B_0}{2} \int \frac{d(x^2+px+q)}{(x^2+px+q)^n} + \left(C_0 - \frac{B_0p}{2}\right) \int \frac{d\left(x + \frac{p}{2}\right)}{\left(\left(x + \frac{p}{2}\right)^2 + \left(\frac{\sqrt{4q-p^2}}{2}\right)^2\right)^n} = \\ &= \frac{B_0}{2} \cdot \int (x^2+px+q)^{-n} d(x^2+px+q) + \frac{2C_0 - B_0p}{2} \int \frac{dt}{(t^2+a^2)^n} = \\ &= \frac{B_0}{2} \cdot \frac{(x^2+px+q)^{-n+1}}{-n+1} + \frac{2C_0 - B_0p}{2} \int \frac{dt}{(t^2+a^2)^n} = \\ &= \frac{B_0}{2(1-n)(x^2+px+q)^{n-1}} + \frac{2C_0 - B_0p}{2} \int \frac{dt}{(t^2+a^2)^n}, \end{aligned}$$

bu yerda $t = x + \frac{p}{2}, \quad a = \frac{\sqrt{4q-p^2}}{2}.$

Oxiridan ko‘rinadiki, agar $J_m = \int \frac{dt}{(t^2 + a^2)^m}$ ($m \in N$), ko‘rinishdagi integrallarni ololsak, masala haldir. Agar bu integralda $m=1$ desak, $J_1 = \int \frac{dx}{t^2 + a^2} = \frac{1}{a} \arctg \frac{t}{a} + C$ - ma’lum.

Endi, $2 \leq m \in N$ bo‘lgan holni qaraylik.

$$J_m = \int \frac{dt}{(t^2 + a^2)^m} = \frac{1}{a^2} \int \frac{(t^2 + a^2) - t^2}{(t^2 + a^2)^m} dt =$$

$$\frac{1}{a^2} \left(\int \frac{dt}{(t^2 + a^2)^{m-1}} - \int \frac{t^2 dt}{(t^2 + a^2)^m} \right) = \frac{1}{a^2} \left(J_{m-1} - \int \frac{t^2 dt}{(t^2 + a^2)^m} \right)$$

Oxirgi integralni bo‘laklaymiz: $u = t, dv = \frac{t dt}{(t^2 + a^2)^m} \Rightarrow du = dt,$

$$v = \frac{1}{2(1-m)(t^2 + a^2)^{m-1}} = -\frac{1}{2(m-1)(t^2 + a^2)^{m-1}}.$$

$$J_m = \frac{1}{a^2} \left(J_{m-1} - \left(-\frac{t}{2(m-1)(t^2 + a^2)^{m-1}} - \int \left(-\frac{1}{2(m-1)(t^2 + a^2)^{m-1}} \right) dt \right) \right) =$$

$$= \frac{1}{a^2} \left(J_{m-1} + \frac{t}{2(m-1)(t^2 + a^2)^{m-1}} - \frac{1}{2(m-1)} J_{m-1} \right) =$$

$$= \frac{t}{2a^2(m-1)(t^2 + a^2)^{m-1}} + \frac{2m-3}{2a^2(m-1)} J_{m-1}.$$

$$\text{Demak, } J_m = \frac{t}{2a^2(m-1)(t^2 + a^2)^{m-1}} + \frac{2m-3}{2a^2(m-1)} J_{m-1};$$

$$J_m = \frac{1}{2a^2(m-1)} \left(\frac{t}{(t^2 + a^2)^{m-1}} + (2m-3)J_{m-1} \right), \quad m = 2, 3, \dots$$

rekurrent (qaytma) formulani olamiz.

Endi, $m=2, 3, \dots, n$ qiymatlarni berish natijasida J_n ni topamiz.

Shunday qilib, berilgan ratsional kasrning, yuqoridagi usullar bilan, integralini topa olamiz, ya’ni ratsional kasr integrali elementar funksiyadan iborat bo‘lar ekan.

3-misol. $\int \frac{x^6 + 1}{x^5 + x^2} dx$ integralni toping.

Yechish: $\frac{x^6 + 1}{x^5 + x^2} = \frac{(x^6 + x^3) + 1 - x^3}{x^5 + x^2} = \frac{x^3(x^3 + 1) + (1 - x^3)}{x^2(x^3 + 1)} = x + \frac{1 - x^3}{x^2(x^3 + 1)}$

$$\int \frac{x^6 + 1}{x^5 + x^2} dx = \int \left(x + \frac{1 - x^3}{x^2(x^3 + 1)} \right) dx = \frac{x^2}{2} + \int \frac{1 - x^3}{x^2(x+1)(x^2 - x + 1)} dx.$$

$$\frac{1 - x^3}{x^2(x+1)(x^2 - x + 1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x+1} + \frac{A_4x + A_5}{x^2 - x + 1};$$

$$1 - x^3 = A_1x(x^3 + 1) + A_2(x^3 + 1) + A_3x^2(x^2 - x + 1) + (A_4x + A_5)x^2(x + 1);$$

$$1 - x^3 = A_1x^4 + A_1x + A_2x^3 + A_2 + A_3x^4 - A_3x^3 + A_3x^2 + A_4x^4 + A_4x^3 + A_5x^3 + A_5x^2;$$

$$\left\{ \begin{array}{l} A_1 + A_3 + A_4 = 0, \\ A_2 - A_3 + A_4 + A_5 = -1, \\ A_3 + A_5 = 0, \\ A_1 = 0, \\ A_2 = 1; \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} A_1 = 0, \\ A_2 = 1, \\ A_3 + A_4 = 0, \\ -A_3 + A_4 + A_5 = -2, \\ A_3 + A_5 = 0; \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} A_1 = 0, \\ A_2 = 1, \\ A_4 = -A_3, \\ -A_3 - A_3 - A_3 = -2, \\ A_5 = -A_3; \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} A_1 = 0, \\ A_2 = 1, \\ A_3 = \frac{2}{3}, \\ A_4 = -\frac{2}{3}, \\ A_5 = -\frac{2}{3}. \end{array} \right.$$

$$\begin{aligned} \frac{1-x^3}{x^2(x^3+1)} &= \frac{1}{x^2} + \frac{2}{3} \left(\frac{1}{x+1} - \frac{x+1}{x^2-x+1} \right) \\ \int \frac{x^6+1}{x^5+x^2} dx &= \frac{x^2}{2} + \int \left(\frac{1}{x^2} + \frac{2}{3} \left(\frac{1}{x+1} - \frac{x+1}{x^2-x+1} \right) \right) dx = \\ &= \frac{x^2}{2} - \frac{1}{x} + \frac{2}{3} \left(\ln|x+1| - \int \frac{x+1}{x^2-x+1} dx \right) = \\ &= \frac{x^2}{2} - \frac{1}{x} + \frac{2}{3} \left(\ln|x+1| - \int \frac{\frac{1}{2}(2x-1) + \frac{3}{2}}{x^2-x+1} dx \right) = \\ &= \frac{x^2}{2} - \frac{1}{x} + \frac{2}{3} \left(\ln|x+1| - \frac{1}{2} \ln(x^2-x+1) - \frac{3}{2} \cdot \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} \right) + C = \\ &= \frac{x^2}{2} - \frac{1}{x} + \frac{2}{3} \ln|x+1| - \frac{1}{3} \ln(x^2-x+1) - \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} + C = \\ &= \frac{x^2}{2} - \frac{1}{x} + \frac{1}{3} \ln \frac{x^2+2x+1}{x^2-x+1} - \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} + C. \end{aligned}$$

12.4. Ba'zi bir trigonometrik ifodalarni integrallash

1. $\int R(\sin x; \cos x) dx$ integralda $R(u; v)$ o'z argumentlarining ratsional funksiyasi bo'lsin. U holda, bu integralda umumiy trigonometrik almashtirish deb ataluvchi $\operatorname{tg} \frac{x}{2} = t \Rightarrow x = 2 \operatorname{arctg} t$ almashtirish yordamida ratsional funksiya integraliga kelinadi. Haqiqatdan ham,

$$\sin x = \frac{2 \operatorname{tg} \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} = \frac{2t}{1+t^2}, \quad \cos x = \frac{1 - \operatorname{tg}^2 \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2dt}{1+t^2} \quad \text{ekanligini}$$

e'tiborga olsak,

$$\int R(\sin x; \cos x) dx = \int R\left(\frac{2t}{1+t^2}; \frac{1-t^2}{1+t^2}\right) \cdot \frac{2}{1+t^2} dt = \int r(t) dt,$$

bu yerda $r(t) = \frac{2}{1+t^2} R\left(\frac{2t}{1+t^2}; \frac{1-t^2}{1+t^2}\right)$ - ratsional funksiya.

Masalan, bu almashtirish yordamida integrallar jadvalidagi 16-formulani keltirib chiqarish mumkin:

$$\int \frac{du}{\sin x} = \int \frac{\frac{2}{1+t^2} dt}{\frac{2t}{1+t^2}} = \int \frac{dt}{t} = \ln |t| + C = \ln \left| \operatorname{tg} \frac{x}{2} \right| + C$$

17-formula esa $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ ekanligidan va 16-formuladan kelib chiqadi.

2. $\int R(\sin x) \cos x dx$ integralda $R(u)$ –ratsional funksiya.

Bu integral uchun ham yuqoridagi umumiy almashtirishni qilib ratsional funksiya integraliga kelish mumkin. Ammo, bu yerda $\sin x = t$ almashtirish (o'rniga qo'yish) qulayroq, chunki $\cos x dx = d(\sin x)$ dir.

$$\int R(\sin x) \cos x dx = \int R(t) dt$$

3. $\int R(\cos x) \sin x dx$, $R(u)$ –ratsional funksiya.

Bu yerda ham $\operatorname{tg} \frac{x}{2} = t$ umumiy almashtirishdan yoki qulayroq bo'lgan $\cos x = t$ dan foydalanish mumkin.

4. $\int R(\operatorname{tg} x) dx$, $R(u)$ –ratsional funksiya.

$\operatorname{tg} \frac{x}{2} = t$ umumiy almashtirishdan foydalanish mumkin. Ammo qulayroq bo'lgan

$\operatorname{tg} x = t \Rightarrow x = \arctg t \Rightarrow dx = \frac{dt}{1+t^2}$ almashtirishdan foydalansak,

$$\int R(\operatorname{tg} x) dx = \int R(t) \frac{1}{1+t^2} dt = \int \frac{R(t)}{1+t^2} dt = \int r(t) dt \quad \text{ga kelamiz, bu yerda } r(t) = \frac{R(t)}{1+t^2} -$$

ratsional funksiya.

5. Sinus va kosinusning toq darajalarini integrallash

$$\begin{aligned} a) \int \sin^{2n+1} x dx &= \int (\sin^2 x)^n \cdot \sin x dx = -\int (1 - \cos^2 x)^n d \cos x = \\ &= -\int (1 - t^2)^n dt, \quad \text{bu yerda } t = \cos x. \end{aligned}$$

$$\begin{aligned} b) \int \cos^{2n+1} x dx &= \int (\cos^2 x)^n \cdot \cos x dx = \int (1 - \sin^2 x)^n d \sin x = \\ &= \int (1 - t^2)^n dt, \quad \text{bu yerda } t = \sin x. \end{aligned}$$

6. Sinus va kosinusning juft darajalarini integrallash.

Bunday integralda $\sin^2 x = \frac{1 - \cos 2x}{2}$, $\cos^2 x = \frac{1 + \cos 2x}{2}$ darajani pasaytirish formulalaridan foydalanish mumkin:

$$a) \int \sin^{2m} x dx = \int (\sin^2 x)^m dx = \int \left(\frac{1 - \cos 2x}{2} \right)^m dx = \frac{1}{2^m} \int (1 - \cos 2x)^m dx;$$

$$b) \int \cos^{2m} x dx = \frac{1}{2^m} \int (1 + \cos 2x)^m dx.$$

$$\begin{aligned}
 \text{Masalan, } \int \sin^4 x dx &= \frac{1}{2^2} \int (1 - \cos 2x)^2 dx = \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) dx = \\
 &= \frac{1}{4} \left(x - \sin 2x + \int \cos^2 2x dx \right) = \frac{1}{4} \left(x - \sin 2x + \frac{1}{2} \int (1 + \cos 4x) dx \right) = \\
 &= \frac{1}{4} \left(x - \sin 2x + \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right) \right) + C = \frac{1}{4} \left(\frac{3}{2} x - \sin 2x + \frac{1}{8} \sin 4x \right) + C.
 \end{aligned}$$

7. Asosiy trigonometrik funksiyalarning ixtiyoriy butun ko'rsatkichli darajalarini integrallash.

a) $S_n = \int \sin^n x dx$ $n \in \mathbb{Z}$ deyilik.

Bu integralda $n = -2; -1; 0; 1$ bo'lsa, jadval integrallarini olamiz, ya'ni

$$\begin{aligned}
 S_{-2} &= \int \sin^{-2} x dx = \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + c \\
 S_{-1} &= \int \sin^{-1} x dx = \int \frac{dx}{\sin x} = \ln \left| \operatorname{tg} \frac{x}{2} \right| + C \\
 S_0 &= \int \sin^0 x dx = \int dx = x + C \\
 S_1 &= \int \sin x dx = -\cos x + C
 \end{aligned} \tag{12.4.1}$$

Aytaylik, n ning qiymati bu ko'rilgan qiymatlardan farq qilsin, u vaqtda,

$S_n = \int \sin^{n-1} x \cdot \sin x dx$ ni bo'laklaymiz:

$$u = \sin^{n-1} x, \quad dv = \sin x dx \Rightarrow du = (n-1) \sin^{n-2} x \cdot \cos x dx, \quad v = -\cos x;$$

$$\begin{aligned}
 S_n &= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \cdot \cos^2 x dx = \\
 &= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \cdot (1 - \sin^2 x) dx = \\
 &= -\sin^{n-1} x \cdot \cos x + (n-1) S_{n-2} - (n-1) S_n.
 \end{aligned}$$

Buni S_n ga nisbatan yechib,

$$S_n = \frac{n-1}{n} S_{n-2} - \frac{1}{n} \sin^{n-1} x \cos x, \quad n=2,3,\dots \tag{12.4.2}$$

S_{n-2} ga nisbatan yechib esa,

$$S_{n-2} = \frac{n}{n-1} S_n + \frac{1}{n-1} \sin^{n-1} x \cos x, \quad n=-1,-2,\dots \tag{12.4.3}$$

rekurrent formulalarni olamiz. Bu (12.4.2) va (12.4.3) lar yordamida (12.4.1) ni hisobga olgan holda $\forall n \in \mathbb{Z}$ uchun J_n integrallarni topa olamiz.

b) $K_n = \int \cos^n x dx$, $(n \in \mathbb{Z})$ bo'lsin.

Bu integral $n = -2; -1; 0; 1$ bo'lganda jadval integrallaridir, ya'ni

$$K_{-2} = \operatorname{tg} x + c, \quad K_{-1} = \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C, \quad K_0 = x + C, \quad K_1 = \sin x + C \tag{12.4.4}$$

Endi n ning boshqa butun qiymatlarini qaraymiz. Oldingi banddagidek ishlarni bajarib,

$$K_n = \frac{n-1}{n} K_{n-2} + \frac{1}{n} \cos^{n-1} x \cdot \sin x \quad n = 2; 3; \dots \tag{12.4.5}$$

$$K_{n-2} = \frac{n}{n-1} K_n - \frac{1}{n-1} \cos^{n-1} x \cdot \sin x \quad n = -1; -2; \dots \quad (12.4.6)$$

rekurrent formulalarni olamiz.

$$\text{c) } T_n = \int tg^n x dx \quad (= \int ctg^{-n} x dx) \quad (n \in Z) \text{ deylik}$$

$$T_{-1} = \ln |\sin x| + C, \quad T_0 = x + C, \quad T_1 = -\ln |\cos x| + C \quad (12.4.7)$$

jadval integrallaridir.

$$T_n = \int tg^{n-2} x \cdot tg^2 x dx = \int tg^{n-2} x \cdot \left(-1 + \frac{1}{\cos^2 x}\right) dx = -T_{n-2} + \frac{1}{n-1} tg^{n-1} x$$

Bundan

$$T_n = -T_{n-2} + \frac{1}{n-1} tg^{n-1} x \quad n=2,3,\dots \quad (12.4.8)$$

yoki

$$T_{n-2} = -T_n + \frac{1}{n-1} tg^{n-1} x \quad n=0,-1,-2,\dots \quad (12.4.9)$$

rekurrent formulalarni olamiz.

Misollar. 1) $\int \frac{dx}{\sin^3 x} = J_{-3} = \frac{-1}{-2} J_{-1} + \frac{1}{-2} \sin^{-2} x \cdot \cos x =$

$$= \frac{1}{2} \left(J_{-1} - \frac{\cos x}{\sin^2 x} \right) = \frac{1}{2} \left(\ln \left| \frac{x}{2} \right| - \frac{\cos x}{\sin^2 x} \right) + C$$

2) $\int \frac{dx}{\cos^4 x} = K_{-4} = \frac{-2}{-3} K_{-2} - \frac{1}{-3} \cos^{-3} x \cdot \sin x = \frac{1}{3} \left(2tgx + \frac{\sin x}{\cos^3 x} \right) + C =$

$$= \frac{1}{3} (2tgx + tgx \cdot (1 + tg^2 x)) + C = \frac{1}{3} (2tgx + tgx + tg^3 x) + C = \frac{1}{3} (3tgx + tg^3 x) + C = tgx + \frac{1}{3} tg^3 x + C$$

3) $\int ctg^{-3} x dx = \int tg^3 x dx = T_3 = -T_1 + \frac{1}{2} tg^2 x = +\ln |\cos x| + \frac{1}{2} tg^2 x + C$

12.5. Ba'zi bir irratsional ifodalarni integrallash

$$1. \int R \left[x; \left(\frac{ax+b}{kx+l} \right)^{\frac{m_1}{n_1}}; \dots; \left(\frac{ax+b}{kx+l} \right)^{\frac{m_p}{n_p}} \right] dx \quad \text{integralda } P(x; u_1; \dots; u_p) \text{ o'z argumentlarining}$$

ratsional funksiyasi a, b, k, l, m_i, n_i lar berilgan haqiqiy sonlar bo'lib $|k| + |l| > 0$, $m_i \in Z$, $n_i \in N$ ($i=1; 2; \dots$) hamda $al - bk \neq 0$ deb faraz qilamiz.

Bu integralda $\frac{ax+b}{kx+l} = t^\lambda$ almashtirish qilamiz, bu yerda λ berilgan $n_1, n_2, \dots, n_p$ natural sonlarning eng kichik karraliligidir.

Almashtirishni x ga nisbatan yechib,

$$x = \frac{lt^\lambda - B}{a - kt^\lambda} = \varphi(t)$$

ni olamiz.

Tenglikning o'ng tomoni t ga nisbatan ratsional funksiya ekanligi ravshandir. Uni differensiallab,

$$dx = \varphi'(t) dt$$

ni olamiz. Ratsional kasrning hosilasi ham ratsional kasr ekanligidan $\varphi'(t)$ ham ratsional funksiyadir.

Endi, almashtirishni integralga qo'yib,

$$\int R\left[x; \left(\frac{ax+b}{kx+l}\right)^{\frac{m_1}{n_1}}; \dots; \left(\frac{ax+b}{kx+l}\right)^{\frac{m_p}{n_p}}\right] dx = \int R[\varphi(t); t^{\mu_1}; \dots; t^{\mu_p}] \varphi'(t) dt = \int r(t) dt$$

ni olamiz. Bu yerda $r(t) = R[\varphi(t); t^{\mu_1}; \dots; t^{\mu_p}] \varphi'(t)$ bo'lib, u ratsional funksiyadan iboratdir, chunki

$$\mu_i = \frac{m_i}{n_i} \lambda \in \mathbb{Z}.$$

Misol. $\int \frac{\sqrt{x+1}}{1+\sqrt[3]{x+1}}$ ni toping.

Yechish: $\sqrt{x+1} = (x+1)^{\frac{1}{2}}; \sqrt[3]{x+1} = (x+1)^{\frac{1}{3}} \Rightarrow \lambda = 6; x+1 = t^6; x = t^6 - 1, dx = 6t^5 dt$

$$\begin{aligned} \int \frac{\sqrt{x+1}}{1+\sqrt[3]{x+1}} dx &= \int \frac{t^3}{1+t^2} \cdot 6t^5 dt = 6 \int \frac{t^8}{t^2+1} dt = 6 \int \left(t^6 - t^4 + t^2 - 1 + \frac{1}{1+t^2} \right) dt = \\ &= 6 \left(\frac{t^7}{7} - \frac{t^5}{5} + \frac{t^3}{3} - t + \arctgt \right) + C = 6 \left(\left(\frac{t^6}{7} - \frac{t^4}{5} + \frac{t^2}{3} - 1 \right) t + \arctgt \right) + C = \\ &= 6 \left(\left(\frac{x+1}{7} - \frac{\sqrt[3]{(x+1)^2}}{5} + \frac{\sqrt[3]{x+1}}{3} - 1 \right) \sqrt{x+1} + \arctg \sqrt[6]{x+1} \right) + C \end{aligned}$$

2. $\int R(x; \sqrt{ax^2+bx+c}) dx$, $R(x;u)$ – o'z argumentlarining ratsional funksiyasi, $a \neq 0$.

Bu integralni olishda trigonometrik almashtirishlardan yoki Eyler almashtirishlaridan foydalanish mumkin. Bu yerda Eyler almashtirishlarini keltiramiz.

1) Agar $a > 0$ bo'lsa,

$$\sqrt{ax^2+bx+c} = t \pm \sqrt{ax} \quad (12.5.1)$$

Eylerning birinchi almashtirishini qilib,

$$x = \frac{t^2 - c}{b \mp 2\sqrt{at}} = \varphi(t) \quad (12.5.2)$$

ni olamiz. Bundan

$$dx = \frac{2bt \mp 2\sqrt{at}^2 + \mp 2\sqrt{at}}{(b \mp 2\sqrt{at})^2} dt = \varphi'(t) dt \quad (12.5.3)$$

ni (12.5.2) ni (12.5.1) ga qo'yib,

$$\sqrt{ax^2+bx+c} = \frac{bt \mp \sqrt{at}^2 \mp \sqrt{ac}}{b \mp 1\sqrt{at}} = \psi(t) \quad (12.5.4)$$

ni olamiz.

(12.5.2), (12.5.3) va (12.5.4) larni integralga qo'yib,

$$\int R(x; \sqrt{ax^2+bx+c}) dx = \int R[\varphi(t); \psi(t)] \cdot \varphi'(t) dt = \int r(t) dt$$

ga ega bo'lamiz, bu yerda $r(t) = R[\varphi(t); \psi(t)] \cdot \varphi'(t)$ funksiya $\varphi(t)$, $\psi(t)$ va $R(x;u)$ – ratsional funksiyalar bo'lganligi sababli ratsional kasrdan iborat bo'ladi.

Almashtirishdagi «+» va «-» ishoralardan ixtiyoriy birini tanlash mumkin.

Misol tariqasida integrallar jadvalidagi 15-formulani keltirib chiqaraylik.

$$\int \frac{dx}{\sqrt{x^2 \pm q^2}}, \quad a=1>0.$$

(12.5.1) da «-» ishorani tanlaymiz: $\sqrt{x^2 \pm q^2} = t - x \Rightarrow t = x + \sqrt{x^2 \pm q^2}$;

$$x^2 \pm q^2 = t^2 - 2tx + x^2 \Rightarrow x = \frac{t^2 \mp q^2}{2t} = \frac{1}{2} \left(t \mp \frac{q^2}{t} \right);$$

$$dx = \frac{1}{2} \left(1 \pm \frac{q^2}{t^2} \right) dt = \frac{t^2 \pm q^2}{2t^2} dt; \quad \sqrt{x^2 \pm q^2} = t - \frac{t^2 \mp q^2}{2t} = \frac{t^2 \pm q^2}{2t}$$

$$\int \frac{dx}{\sqrt{x^2 \pm q^2}} = \int \frac{\frac{t^2 \pm q^2}{2t^2} dt}{\frac{t^2 \pm q^2}{2t}} = \int \frac{dt}{t} = \ln |t| + C = \ln \left| x + \sqrt{x^2 \pm q^2} \right| + C.$$

1-eslatma. Almashtirish yordamida berilgan integralni ratsional funksiyaning integraliga keltirish uni *ratsionallashtirish* deb ataladi.

2) Agar $c>0$ bo'lsa,

$$\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c}$$

Eylarning ikkinchi almashtirishni qilib, yuqoridagiga o'xshash ishlar bajarilsa, integral ratsionallashtiriladi.

3) Agar ildiz ostidagi kvadrat uchhad diskriminanti musbat bo'lsa, ya'ni u ikkita ildizga ega bo'lsa,

$$\sqrt{ax^2 + bx + c} = t(x - \alpha)$$

Eylarning uchinchi almashtirishni qo'llash mumkin, bu yerda α kvadrat uchhad ildizlaridan biri. Bu almashtirishda ham 1) banddagiga o'xshash ishlarni bajarib, integral ratsionallashtiriladi.

4) Agar kvadrat uchhad diskriminanti nolga teng va $a>0$ bo'lsa, ildiz ostidagi ifoda to'la kvadratdan iborat bo'lib qoladi va irratsionallik o'z-o'zidan yo'qoladi, ya'ni bu holda integral ostida irratsionallik bo'lmaydi.

5) Agar kvadrat uchhad diskriminanti musbat bo'lmay $a<0$ bo'lsa, $\sqrt{ax^2 + bx + c}$ - ildiz ma'nosini yo'qotadi va integral ham ma'noga ega bo'lmaydi.

2-eslatma $\sqrt{ax^2 + bx + c}$ - ildiz biror oraliqda ma'noga ega bo'ladigan va kvadrat uchhad diskriminanti noldan farqli bo'lgan holda Eyler almashtirishlaridan birini qo'llash mumkin bo'ladi, albatta.

12.6. Olinmaydigan integrallar haqida

Yuqorida berilgan $f(x)$ funksiyaning boshlang'ich funksiyasi, ya'ni aniqmas integrali elementlar funksiya orqali ifodalanadigan hollarni ko'rdik. Bu yerda boshlang'ich funksiya elementar bo'lmaydigan funksiyalar haqida so'z yuritamiz. Masalan,

$$\int e^{-x^2} dx, \quad \int \frac{\sin x}{x} dx, \quad \int \frac{\cos x}{x} dx, \quad \int \sin x^2 dx, \quad \int \cos x^2 dx$$

integrallarni olsak, ular elementar funksiyalar orqali ifodalanmaydi. Bu va ularga o'xshash integrallarni *olinmaydigan integrallar* deyish odat tusiga kirgan. Shu sababli biz

ham bunday integrallarni, an'anani saqlagan holda, olinmaydigan integrallar deb ataymiz. Shuni ham aytish lozimki, olinmaydigan integral deganda integral mavjud emas ekan deb tushunmaslik kerak, ular mavjuddir, ammo, elementar funksiya orqali ifodalanmay maxsus funksiyalar orqali ifodalanadi. Masalan,

$$\phi(t) = \frac{1}{\sqrt{2n}} \int e^{-\frac{t^2}{2}} dt$$

integral ehtimollar nazariyasida muhim ahamiyatga ega bo'lib, uni *ehtimollik integrali* deb yuritiladi.

$$\int e^{-x^2} dx = \sqrt{\pi} \phi(x\sqrt{2}) + C$$

ekanligini $t = x\sqrt{2}$ almashtirish bilan keltirib chiqarish mumkin, ya'ni yuqoridagi olinmaydigan integral ehtimollik integrali deb ataluvchi maxsus funksiya orqali ifodalanar ekan.

$\int x^m (a + bx^n)^p dx$ integral *binomial differensial integrali* deb atalib, bu yerda m, n, p lar ratsional, a va b lar esa noldan farqli haqiqiy sonlardir. P.L.Chebichev tomonidan, bu integral 1) p - butun son; 2) $\frac{m+1}{n}$ - butun son; 3) $\frac{m+1}{n} + p$ - butun son bo'lgan hollardan biri sodir bo'lgandagina elementar funksiya iborat bo'lishi, ya'ni olinishi isbotlangandir. Boshqa holda u olinmaydigan integraldir.